Engineering Notes

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New Results on the Optimal Spacecraft Attitude Maneuver Problem

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Introduction

JUNKINS and Turner[1] have treated the general problem of optimal large-angle attitude maneuvers of rigid spacecraft by proposing an integral performance index which involves the control torques. The singularity-free formulation, utilizing Euler parameter attitude representation, admits smooth external torques. Closed-form solutions were obtained for special cases of single-axis maneuvers, and the more general three-axis maneuvers (solutions of nonlinear two-point boundary-value problems) were computed numerically by a differential correction scheme in conjunction with a continuation method. Central to this algorithm is the minimization of the norm of the Euler parameter costates.

In this Note, we show that is is not necessary to rigorously minimize the above norm, because the same solution can be obtained more efficiently by constraining the Euler parameter costate vector to be orthogonal to the Euler parameter vector. Furthermore, a novel physical/geometrical interpretation of the Euler parameter costates is demonstrated.

Spacecraft Kinematics and Dynamics

Orientation and Rotational Kinematics

The orientation of an arbitrary body-fixed frame $\{\hat{b}\}$ with respect to an arbitrary inertial frame $\{\hat{n}\}$ is given by

$$\{\hat{b}\} = [C]\{\hat{n}\} \tag{1}$$

Although the direction cosine matrix $[C]$ is commonly parameterized in terms of a set of three Euler angles, it is advantageous[2] to use the set of four Euler parameters $(\beta_0, \beta_1, \beta_2, \beta_3)$ instead. Then

$$[C] = [C(\beta)] =$$

$$\begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix} \tag{2}$$

The Euler parameters have a geometrical interpretation in terms of Euler's theorem, which states that a completely general rotation of a rigid body can be accomplished by a single rotation (through the principal angle ϕ_{nb}) about a line (the principal line $\hat{\ell}_{nb}$) which is fixed relative to both the arbitrary body-fixed axes and the inertial reference frame. The Euler parameters are related to the direction cosines $(\sigma_{nb1}, \ell_{nb2}, \ell_{nb3})$ of $\hat{\ell}_{nb}$ and ϕ_{nb} by

$$\beta_0 = \cos(\phi_{nb}/2)$$

$$\beta_i = \ell_{nb_i}\sin(\phi_{nb}/2), \qquad i=1,2,3 \tag{3}$$

Thus the Euler parameters are once-redundant due to the constraint

$$\sum_{i=0}^{3} \beta_i^2 = 1 \tag{4}$$

The Euler parameter differential equations can be written[3] as

$$\dot{\beta} = [G(\omega)]\beta \tag{5}$$

where ω_i $(i=1,2,3)$ are $\{\hat{b}\}$ components of the angular velocity of $\{\hat{b}\}$ with respect to $\{\hat{n}\}$, and

$$[G(\omega)] = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_3 & -\omega_1 & 0 \end{bmatrix}$$

Notice that

$$[G] = -[G]^T \tag{6}$$

Spacecraft Dynamics

Euler's equations for the rotational motion of a rigid spacecraft in a body-fixed principal axis system are as follows:

$$\dot{\omega}_1 = -\alpha_1\omega_2\omega_3 + u_1/I_1$$

$$\dot{\omega}_2 = -\alpha_2\omega_1\omega_3 + u_2/I_2$$

$$\dot{\omega}_3 = -\alpha_3\omega_3\omega_1\omega_2 + u_3/I_3 \tag{7}$$

where I_1, I_2, and I_3 are the spacecraft principal inertias; u_1, u_2, and u_3 are $\{b\}$ components of the control torque; and $\alpha_1 = (I_3 - I_2)/I_1$, $\alpha_2 = (I_1 - I_3)/I_2$, and $\alpha_3 = (I_2 - I_1)/I_3$.

Optimal Maneuvers

The states of the spacecraft modeled above are β_0, β_1, β_2, β_3, ω_1, ω_2, and ω_3. Maneuvers are performed to transfer the spacecraft from an initial state to a prescribed final state at time T, using external control torques u_1, u_2, and u_3. The

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performance index selected is

$$J=\frac{1}{2}\int_0^T\left[\sum_{i=1}^{3}u_i^2(t)\right]\mathrm{d}t \tag{8}$$

The necessary conditions for optimality are derived via the Pontryagin principle.[4] The Hamiltonian, H for the system is written as

$$H=\frac{1}{2}\sum_{i=1}^{3}u_i^2+\gamma^T\dot{\beta}+\lambda^T\dot{\omega} \tag{9}$$

where γ_i and λ_i are the Lagrange multipliers or costate variables.

State equations:

$$\dot{\beta}=[G(\omega)]\beta \tag{10}$$

$$\dot{\omega}_1=-\alpha_1\omega_2\omega_3-\lambda_1/I_1^2$$

$$\dot{\omega}_2=-\alpha_2\omega_1\omega_3-\lambda_2/I_2^2$$

$$\dot{\omega}_3=-\alpha_3\omega_1\omega_2-\lambda_3/I_3^2 \tag{11}$$

Costate equations:

$$\dot{\gamma}=[G(\omega)]\gamma \tag{12}$$

$$\dot{\lambda}=\begin{bmatrix}0 & \alpha_3\lambda_3 & \alpha_2\lambda_2\\ \alpha_3\lambda_3 & 0 & \alpha_1\lambda_1\\ \alpha_2\lambda_2 & \alpha_1\lambda_1 & 0\end{bmatrix}\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix}\beta_1 & -\beta_0 & -\beta_3 & \beta_2\\ \beta_2 & \beta_3 & -\beta_0 & \beta_1\\ \beta_3 & -\beta_2 & \beta_1 & -\beta_0\end{bmatrix}\begin{Bmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\\ \gamma_3\end{Bmatrix} \tag{13}$$

Boundary conditions at the initial and final times are specified on $\beta(t)$ and $\omega(t)$.

Properties of the State and Costate Differential Equations

Note the following three integrals of the state and costate differential equations

I. $$\beta^T(t)\beta(t)=1 \tag{14}$$

which is well known and follows from the definition of the Euler parameters [Eq. (3)].

II. $$\gamma^T(t)\gamma(t)=\text{const}=B^2 \tag{15}$$

as $\gamma(t)$ satisfies a norm-preserving skew-symmetric differential equation (12) similar to Eq. (10). Hence the γ_i, if normalized by B, represent Euler parameters describing the body orientation with respect to some other inertial frame.

III. $$\beta^T(t)\gamma(t)=\text{const}=C \tag{16}$$

This can be shown by differentiating $\beta^T(t)\gamma(t)$, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\beta^T(t)\gamma(t)]=\dot{\beta}^T(t)\gamma(t)+\beta^T(t)\dot{\gamma}(t) \tag{17}$$

Eliminating $\dot{\beta}^T$ and $\dot{\gamma}(t)$ via Eqs. (10) and (12) results in

$$\frac{\mathrm{d}}{\mathrm{d}t}[\beta^T(t)\gamma(t)]=\beta^T[G^T]\gamma+\beta^T[G]\gamma \tag{18}$$

Using Eq. (7) in Eq. (21) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\beta^T(t)\gamma(t)]=\beta^T[G^T]\gamma+\beta^T(-[G^T])\gamma=0 \tag{19}$$

and hence $\beta^T(t)\gamma(t)=\text{const}=C$.

Admissible Euler Parameter Costates

It has been shown[1] that B, and hence γ, are not unique, i.e., different $\gamma(t)$ give rise to the same optimal control. It is important to note, however, that although an infinite number of nontrivial $\gamma(0)$ could satisfy the necessary conditions for optimality, they are not totally arbitrary. As a simple counterexample, let us consider $\gamma(t)=\rho\beta(t)$, where ρ is an arbitrary constant. Then

$$\gamma^T\dot{\beta}=\rho\beta^T\dot{\beta}=\frac{1}{2}\rho\frac{\mathrm{d}}{\mathrm{d}t}(\beta^T\beta)=\frac{1}{2}\rho\frac{\mathrm{d}}{\mathrm{d}t}(1)=0$$

Observe that this choice for $\gamma(t)$ decouples the attitude and dynamics in the Hamiltonian equation (9), and only the angular velocity of the spacecraft can be controlled; arbitrary terminal attitude specifications cannot be met. Hence, we conclude,

$$\gamma(t)\neq\rho\beta(t) \tag{20}$$

Since all of the admissible $\gamma(t)$ produce the same control $u(t)$, which in turn is dictated by $\lambda(t)$, the differential equations for $\lambda_i(t)$ should be invariant with respect to $\gamma(t)$. Since the problem at hand is nonlinear and a general analytical solution does not exist, we cannot determine the nature of $\gamma(t)$ quantitatively without numerical solutions. Hence, the gain qualitative insight into the problem, we assume that an admissible numerical solution $\gamma(t)$ has been obtained. Let

$$\gamma_a(t)=\gamma(t)+\delta\gamma(t) \tag{21}$$

neighboring costate vector and $\delta\gamma(t)$ be an admissible variation. To identify the nature of the admissible variations we substitute Eq. (21) into Eq. (13) to obtain

$$\begin{bmatrix}\beta_1 & -\beta_0 & -\beta_3 & \beta_2\\ \beta_2 & \beta_3 & -\beta_0 & -\beta_1\\ \beta_3 & -\beta_2 & \beta_1 & -\beta_0\end{bmatrix}\delta\gamma(t)=0 \tag{22}$$

An additional constraint is obtained by substituting Eq. (21) into Eq. (16) as

$$\beta^T(t)[\gamma(t)+\delta\gamma(t)]=C_a \tag{23}$$

where C_a is an arbitrary constant, depending on $\gamma_a(t)$, or

$$\beta^T(t)\delta\gamma(t)=C_a-C \tag{24}$$

Combining Eqs. (22) and (24), we obtain

$$\begin{bmatrix} \beta_1 & -\beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & -\beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & -\beta_0 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \delta\gamma(t) = \left\{ \begin{matrix} 0 \\ 0 \\ 0 \\ C_a - C \end{matrix} \right\} \tag{25}$$

Since the coefficient matrix is orthogonal, the solution to the above equations can be written as

$$\delta\gamma(t) = (C_a - C)\beta(t) = \alpha_a \beta(t) \tag{26}$$

where

$$\alpha_a = C_a - C$$

Hence we have the family of admissible attitude costates given by

$$\gamma_a(t) = \gamma(t) + \alpha_a \beta(t) \tag{27}$$

It is seen from Eqs. (27) and (20) that $\gamma(t)$ is never zero.

Of particular interest is the solution with $\gamma_m(t)$ such that $\|\gamma_m^T \gamma_m\| = B_m^2$ is a minimum.

In order to perform a minimization, we form the inner product

$$\begin{aligned} \gamma_a^T(t)\gamma_a(t) &= \gamma^T(t)\gamma(t) + 2\alpha_a \gamma^T \beta(t) + \alpha_a^2 \\ &= B^2 + 2\alpha_a C + \alpha_a^2 \end{aligned} \tag{28}$$

Since α_a is the parameter in Eq. (28), it is easy to see by differentiation that α_m corresponding to the minimum norm solution is given by

$$\alpha_m = -C \tag{29}$$

and, correspondingly, the inner product $\gamma_m^T \gamma_m$ is given by

$$B_m^2 = \gamma_m^T(t)\gamma_m(t) = B^2 - 2C^2 + C^2 = B^2 - C^2 \tag{30}$$

For the minimum norm solution

$$B_m^2 = B_m^2 - C_m^2$$

or

$$C_m = \beta^T \gamma_m = 0 \tag{31}$$

The algorithm of Junkins and and Turner[1] can be simplified if the constraint equation (31) is used instead of rigorous minimization of $\gamma^T\gamma$.

Conclusions

It has been shown that imposing an orthogonality constraint between the Euler parameter vector and the corresponding costate vector is equivalent to minimizing the norm of the Euler parameter costate vector. The normalized Euler parameter costates are shown to be Euler parameters, describing the body orientation with respect to a different inertial frame.

The results presented are purely kinematic. They hold for optimal maneuvers for which motion of the reference body axis, with respect to an inertial frame, is decribed by Euler parameters, including certain cases of flexible bodies, multiple rigid bodies, etc. The results neither depend on the form of Euler's equations nor on the performance index, as long as the Euler parameters do not appear in them.

References

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An Exponentially Fitted Adams Method of Numerical Integration

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Nomenclature

$a_0,\ldots,a_k$	= constants of the approximating polynomial (in time) in conventional Adams methoda
$C_0,C_1,C_2,C_3,C_4,$ $C_5,C_6,C_7,$	= constants of approximating function in EFAM
DPE	= drift probable error, equal to 0.6745 of the standard deviation in drift
K_1,K_2,K_3,K_4	= complex constants representing the initial magnitude and orientation of the nutation, precession, trim, and yaw of repose vectors in the complex angle-of-atack plane for a nonrolling coordinate system
$\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3,\mathcal{K}_4$	= complex constants representing the initial magnitude and orientation of the nutation, precession, trim angle, and yaw of repose vectors in the complex angle-of-attack plane for a body-fixed coordinate system

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